SUPPLEMENTARY INFORMATION

Geometrodynamics of Spinning Light

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Here we provide calculations underlying the theory of geometrodynamical evolution of polarized light in smooth inhomogeneous medium and along a curved reflecting surface. All the equations presented in the article are consistently derived starting from Maxwell equations.

1. Diagonalization of Maxwell equations, spin-orbit coupling of photons, Berry phase, and equations of motion

Maxwell equations for monochromatic electric field $\mathbf{E}$ in an inhomogeneous dissipationless dielectric medium can be written as a three-component vector equation

$$-\lambda\!\!\!^{-}_0^2 \nabla\times\left(\nabla\times\mathbf{E}\right)+n^2\mathbf{E}=0\,,$$

or,

$$\left(\lambda\!\!\!^{-}_0^2\nabla^2+n^2\right)\mathbf{E}-\lambda\!\!\!^{-}_0^2\nabla\left(\nabla\mathbf{E}\right)=0\,. \qquad \text{(S1)}$$

where $n^2\left(\mathbf{r}\right)=\varepsilon\left(\mathbf{r}\right)$ is the dielectric constant of the medium. Equation (S1) resembles the Helmholz equation, except for the last, polarization term, which mixes internal and external degrees of freedom of the wave and makes Eq. (S1) non-diagonal[S1,19]. This polarization term in Eq. (S1) guarantees that $\nabla\left(n^2\mathbf{E}\right)=0$ which, in turn, ensures that in a smoothly inhomogeneous medium the wave electric field remains nearly *transverse* with respect to the current momentum $\mathbf{p}$:

$$\mathbf{E} = \mathbf{E}_{\perp} + E_{\|}\mathbf{t}, \ \mathbf{E}_{\perp} \perp \mathbf{t}, \ E_{\|} \sim \mu E_{\perp}. \tag{S2}$$

Here $\mathbf{t} = \mathbf{p}/p$ is the unit vector tangent to the zero-order ray trajectory (S2), $E_{\|}$ is the longitudinal component of the field, and $\mathbf{E}_{\perp}$ is the projection of the electric field on the plane orthogonal to $\mathbf{t}$.

The wave polarization is essentially determined by the transverse field components, $\mathbf{E}_{\perp}$. Hence, the dimension of the problem can be reduced to 2 by *projecting* Maxwell equation (S1) onto the plane orthogonal to $\mathbf{t}$, which eliminates the longitudinal field component $E_{\|}$ from the problem. This implies a description of the wave evolution in a coordinate frame with basis vectors $(\mathbf{v}, \mathbf{w}, \mathbf{t})$ attached to the local direction of momentum, $\mathbf{t}$, Fig. 1A. Vectors $(\mathbf{v}, \mathbf{w})$ provide a natural basis of linear polarizations:

$$\mathbf{E}_{\perp} = E_{\mathrm{v}}\mathbf{v} + E_{\mathrm{w}}\mathbf{w}. \tag{S3}$$

However, the coordinate frame $(\mathbf{v}, \mathbf{w}, \mathbf{t})$ is *non-inertial* in the generic case; it experiences rotation as $\mathbf{t}$ varies along the ray trajectory in an inhomogeneous medium. Such rotation is described by a precession of the triad $(\mathbf{v}, \mathbf{w}, \mathbf{t})$ with some angular velocity $\mathbf{\Lambda}$:

$$\dot{\mathbf{v}} = \mathbf{\Lambda} \times \mathbf{v}, \ \dot{\mathbf{w}} = \mathbf{\Lambda} \times \mathbf{w}, \ \dot{\mathbf{t}} = \mathbf{\Lambda} \times \mathbf{t},$$

$$\mathbf{\Lambda} = (\dot{\mathbf{v}}\mathbf{w})\mathbf{t} + (\dot{\mathbf{w}}\mathbf{t})\mathbf{v} + (\dot{\mathbf{t}}\mathbf{v})\mathbf{w} = \Lambda_{\|}\mathbf{t} + \mathbf{t} \times \dot{\mathbf{t}}, \tag{S4}$$

where $\Lambda_{\|} = \mathbf{\Lambda}\mathbf{t} = \dot{\mathbf{v}}\mathbf{w}$ is the longitudinal component of $\mathbf{\Lambda}$.

When performing a transition to the non-inertial frame $(\mathbf{v}, \mathbf{w}, \mathbf{t})$, effective inertia terms appear in Maxwell equations (S1). Similarly to classical mechanics, they can be derived via the substitution[34] $\frac{\partial \mathbf{E}}{\partial t} \to \frac{\partial \mathbf{E}}{\partial t} + \frac{c}{n} \mathbf{\Lambda} \times \mathbf{E}$, or, $\omega \to \omega + i\frac{c}{n}\mathbf{\Lambda}\times\mathbf{E}$, in Eqs. (S1). [Here the wave velocity c/n occurs because we defined the angular velocity (S4) with respect to the ray length l rather than time.] Neglecting higher-order terms proportional to Λ^2 and $\dot{\Lambda}$, we arrive at

$$\left(\lambda\!\!\!^{-}_0^2\nabla^2 + n^2\right)\mathbf{E} + 2in\lambda\!\!\!^{-}_0\mathbf{\Lambda}\times\mathbf{E} + \lambda\!\!\!^{-}_0^2\nabla\left(\nabla\mathbf{E}\right) = 0. \quad \text{(S5)}$$

The second term here is the *Coriolis term* caused by the rotation of the ray coordinate frame. It is small: $\lambda\!\!\!^{-}_0\Lambda \sim \mu$, but should be taken into account in the first-order approximation in μ.

Projecting equation (S5) onto the plane $(\mathbf{v},\mathbf{w})$, one can show that $\left[\lambda\!\!\!^{-}_0^2\nabla\left(\nabla\mathbf{E}\right)\right]_\perp \simeq 0$ and $\left(\mathbf{\Lambda}\times\mathbf{E}\right)_\perp \simeq \Lambda_\parallel\left(\mathbf{t}\times\mathbf{E}_\perp\right)$ in the first approximation in μ. Thus, the projection onto the $(\mathbf{v},\mathbf{w})$ plane cancels the polarization term, and we have[34]

$$\left(\lambda\!\!\!^{-}_0^2\nabla^2 + n^2\right)\mathbf{E}_\perp + 2in\lambda\!\!\!^{-}_0\Lambda_\parallel\left(\mathbf{t}\times\mathbf{E}_\perp\right) = 0. \quad \text{(S6)}$$

Equation (S6) is a two-component vector equation which becomes diagonal in the basis of *circular* polarizations. Substituting the field as a superposition of right- and left-hand modes,

$$\mathbf{E}_\perp = E^+\xi + E^-\xi^*,\ \xi = \frac{\mathbf{v}+i\mathbf{w}}{\sqrt{2}},\ E^\pm = E_\mathbf{v} \mp iE_\mathbf{w}, \quad \text{(S7)}$$

we obtain

$$\left(\lambda\!\!\!^{-}_0^2\nabla^2 + n^2\right)E^\sigma + 2n\lambda\!\!\!^{-}_0\sigma\Lambda_\parallel E^\sigma = 0. \quad \text{(S8)}$$

Hereafter $\sigma = \pm 1$ denotes the wave *helicity* indicating the two spin states of photons. Equation (S8) shows that, in the first approximation of geometrical optics, these two states evolve independently, and the zero-order polarization degeneracy is removed by the Coriolis term.

Performing substitution $-i\lambda\!\!\!^{-}_0\nabla \to \mathbf{p}$ in Eq. (S8), we find a characteristic equation which gives the wave Hamiltonian

$$\mathcal{H} = p - n - \lambda\!\!\!^{-}_0 \boldsymbol{\sigma}\boldsymbol{\Lambda} = 0\,. \tag{S9}$$

Here we simplified the characteristic equation in the first approximation in μ and introduced the spin angular momentum per photon (in units of $\hbar$), $\boldsymbol{\sigma} = \sigma\mathbf{t}$, so that $\boldsymbol{\sigma}\boldsymbol{\Lambda} = \sigma\Lambda_{\|}$. As compared to the traditional geometrical optics Hamiltonian[26], Eq. (S9) contains an additional spin term which is equivalent to the Coriolis term of spinning particles in a rotating frame[S2,S3]. The Lagrangian corresponding to the Hamiltonian (S9) takes the form

$$\mathcal{L} = \mathbf{p}\dot{\mathbf{r}} - p + n + \lambda\!\!\!^{-}_0 \boldsymbol{\sigma}\boldsymbol{\Lambda} \equiv \mathcal{L}_0 + \mathcal{L}_{\mathrm{SOI}}\,. \tag{S10}$$

where $\mathcal{L}_0 = n - p + \mathbf{p}\dot{\mathbf{r}}$ is the scalar-approximation Lagrangian, whereas $\mathcal{L}_{\mathrm{SOI}} = \lambda\!\!\!^{-}_0 \boldsymbol{\sigma}\boldsymbol{\Lambda}$ is the Lagrangian describing the *spin-orbit coupling* of photons. It should be noted that the spin-orbit term in the Lagrangian had been known for spinning particles before the Berry phase discovery[S4,S5].

In order to represent the spin-orbit Lagrangian in the Berry-phase form, we notice that the co-moving coordinate frame $(\mathbf{v}, \mathbf{w}, \mathbf{t})$ is attached to the direction of the wave momentum $\mathbf{p}$, and the polarization evolution of the wave is essentially *momentum*-dependent (rotations of the ray coordinate frame are independent of the particular space coordinates, $\mathbf{r}$). Therefore, we can parametrize the basis vectors of the ray coordinate frame as:

$$\mathbf{t} = \mathbf{t}(\mathbf{p}),\ \mathbf{v} = \mathbf{v}(\mathbf{p}),\ \mathbf{w} = \mathbf{w}(\mathbf{p})\,. \tag{S11}$$

The transition from the l-parametrization to the $\mathbf{p}$-parametrization is performed via the substitution $\frac{d}{dl} \to \frac{d\mathbf{p}}{dl}\frac{\partial}{\partial \mathbf{p}}$, and the spin-orbit Lagrangian (S10) takes the form of Eq. (1):

$$\mathcal{L}_{\mathrm{SOI}} = -\lambda\!\!\!^{-}_0 \sigma \mathbf{A}(\mathbf{p})\dot{\mathbf{p}}\,, \tag{S12}$$

where

$$A_i = \mathbf{v}\frac{\partial \mathbf{w}}{\partial p_i} = -i\boldsymbol{\xi}^* \frac{\partial \boldsymbol{\xi}}{\partial p_i} \tag{S13}$$

is the *Berry connection* or the *Berry gauge field*.

The Berry connection relates the wave polarization $\mathbf{e} = \mathbf{E}_\perp / E_\perp$ in the neighboring points $\mathbf{p}$ and $\mathbf{p} + d\mathbf{p}$ of momentum space. Note that the polarization evolution depends only on the *direction* of momentum, $\mathbf{t} = \mathbf{p}/p$. Therefore, the evolution in the $\mathbf{p}$ space can be projected onto the unit sphere $\mathrm{S}^2 = \{\mathbf{t}\}$ in momentum space. In this manner, the polarization vector $\mathbf{e}$ is tangent to this sphere, and the Berry connection determines the natural *parallel transport* of $\mathbf{e}$ over the S^2 sphere[2,29–33,S6,S7], Fig. 2A.

The curvature tensor corresponding to the connection (S13) is $F_{ij} = \frac{\partial A_j}{\partial p_i} - \frac{\partial A_i}{\partial p_j}$ which yields

$$F_{ij} = \frac{\partial \mathbf{v}}{\partial p_i}\frac{\partial \mathbf{w}}{\partial p_j} - \frac{\partial \mathbf{v}}{\partial p_j}\frac{\partial \mathbf{w}}{\partial p_i} = -i\left(\frac{\partial \boldsymbol{\xi}^*}{\partial p_i}\frac{\partial \boldsymbol{\xi}}{\partial p_j} - \frac{\partial \boldsymbol{\xi}^*}{\partial p_i}\frac{\partial \boldsymbol{\xi}}{\partial p_j}\right). \tag{S14}$$

This the *Berry curvature* or the *Berry field strength*. It is an antisymmetric tensor which can be characterized by the dual vector $\mathbf{F} = \frac{\partial}{\partial \mathbf{p}} \times \mathbf{A}$, $F_{ij} = \varepsilon_{ijk} F_k$. For electromagnetic waves the Berry curvature takes the form of the "magnetic monopole" Eq. (2)[2,9,10,31,S1]:

$$\mathbf{F} = \frac{\mathbf{p}}{p^3}. \tag{S15}$$

Hence, on the surface of the unit $\mathbf{t}$-sphere (i.e., at $p = 1$) it equals $\mathbf{F} = \mathbf{t}$ indicating the unit Gaussian curvature of the sphere surface.

Note that gauge properties of the potential $\mathbf{A}$ and field $\mathbf{F}$ are directly related to the choice of the co-moving frame $(\mathbf{v}, \mathbf{w}, \mathbf{t})$, which is determined up to an arbitrary rotation about $\mathbf{t}$. Such a local rotation of the coordinate frame on an angle $\alpha = \alpha(\mathbf{p})$, induces the gauge transformation of the basic vector of circular polarizations, $\boldsymbol{\xi}$, Eq. (S7):

$$\boldsymbol{\xi} \to \exp(-i\alpha)\boldsymbol{\xi}, \tag{S16}$$

i.e. $\mathrm{SO}(2)$ rotation of $(\mathbf{v},\mathbf{w})$ is equivalent to $\mathrm{U}(1)$ gauge transformation of $\boldsymbol{\xi}$ (see M. V. Berry in Ref. 2). In turn, the gauge transformation (S16) generates the transformation of the Berry connection (S13) but does not influence the Berry curvature (S14):

$$\mathbf{A} \to \mathbf{A} - \frac{\partial \alpha}{\partial \mathbf{p}}, \ \mathbf{F} \to \mathbf{F}. \tag{S17}$$

Therefore, all the physical quantities which are independent of the coordinate frame (e.g., ray trajectories), should depend on the Berry curvature $\mathbf{F}$ rather than on the gauge-dependent connection $\mathbf{A}$.

The total phase of the wave, Eq. (3), follows directly from the Lagrangian (S10) with the spin-orbit part in the form of Eq. (S12):

$$\Phi = \lambda\!\!\!^{-}_0{}^{-1} \int_\ell \mathcal{L}\, dl = \lambda\!\!\!^{-}_0{}^{-1} \int_\ell \mathbf{p}\, d\mathbf{r} - \sigma \int_{\Gamma_\ell} \mathbf{A}\, d\mathbf{p}, \tag{S18}$$

where we took into account that $\mathcal{H} = 0$, Eq. (S9). Different representations of the polarization evolution stemming from the Berry phase in (S18) are considered below. The Euler-Lagrange equations with the Lagrangian (S10) and (S12) written in the form $\mathcal{L} = \mathcal{L}(\mathbf{p}, \dot{\mathbf{p}}, \mathbf{r}, \dot{\mathbf{r}})$ and varying independently with respect to $\mathbf{p}$ and $\mathbf{r}$ result in the equations of motion

$$\dot{\mathbf{p}} = \nabla n, \ \dot{\mathbf{r}} = \frac{\mathbf{p}}{p} + \lambda\!\!\!^{-}_0 \sigma \dot{\mathbf{p}} \times \mathbf{F} = \frac{\mathbf{p}}{p} + \lambda\!\!\!^{-}_0 \sigma \frac{\dot{\mathbf{p}} \times \mathbf{p}}{p^3}. \tag{S19}$$

They describe the split ray trajectories of the two circularly polarized eigenmodes $\sigma = \pm 1$ of the problem. The ray equations (5) are obtained from (S19) via substitution $\sigma \to S_3$; they describe the center-of-gravity position of the beam with an arbitrary polarization.

2. Evolution of the polarization of light in different representations

The polarization state of the wave can be described by the unit complex two-component Jones vector in the basis of circular polarization:

$$|\psi\rangle = \begin{pmatrix} e^{+} \\ e^{-} \end{pmatrix}, \ \langle\psi|\psi\rangle = 1 \tag{S20}$$

where $e^{\pm} = E^{\pm}/E_{\perp}$ are the normalized amplitudes of the two modes. The Berry phase, Eqs. (3) and (S18),

$$\Phi_B = -\sigma \int_{\Gamma_\ell} \mathbf{A}\, d\mathbf{p}, \tag{S21}$$

acquired by the circularly polarized components with the opposite signs, indicates the following evolution of the Jones vector:

$$|\psi(l)\rangle = \begin{pmatrix} \exp(-i\Phi_B) & 0 \\ 0 & \exp(+i\Phi_B) \end{pmatrix} |\psi(0)\rangle. \tag{S22}$$

It is easy to see that, in the generic case of an elliptical polarization, this equation describes the turn of the polarization ellipse on angle Φ_B with its eccentricity conserved, Fig. 1A.

The differential form of Eq. (S22) is:

$$|\dot{\psi}\rangle = -i(\mathbf{A}\dot{\mathbf{p}})\hat{\sigma}_3|\psi\rangle, \tag{S23}$$

where we use the Pauli matrices $\hat{\vec{\sigma}} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$. Equation (S23) was obtained in Refs. 14,19 and is similar to the equation for the polarization evolution in other spin systems[S8,S9]. Eq. (S23) describes a local *inertia* of the electric field which remains *locally* non-rotating about the ray. Indeed, one can show that Eq. (S23) is equivalent to the equation for the unit electric field vector $\mathbf{e} = \mathbf{E}_\perp / E_\perp$:

$$\dot{\mathbf{e}} = -\left(\mathbf{e}\dot{\mathbf{t}}\right)\mathbf{t}. \qquad \text{(S24)}$$

This is a well-know equation for the parallel transport of vector $\mathbf{e}$ along the ray[2]. According to Eq. (S24), $\mathbf{e}$ does not experience local rotation about $\mathbf{t}$.

As is known, there is an alternative formalism describing the polarization state of light, namely, the Stokes parameters representing the polarization state on the Poincaré sphere. This formalism is quite similar to the quantum mechanical Bloch-sphere representation. Indeed, the Stokes vector is a three-component unit vector defined as

$$\vec{S} = \langle \psi | \hat{\vec{\sigma}} | \psi \rangle,\ \vec{S}^2 = 1. \qquad \text{(S25)}$$

The north and south poles of the Poincaré sphere ($S_3 = \pm 1$) represent the right- and left-hand circular polarizations, whereas the equator ($S_3 = 0$) represents the linear polarizations, Fig. 2B. By differentiating expression (S25) and using Eq. (S23), we find that the Stokes vector obeys the following precession equation[19]:

$$\dot{\vec{S}} = \vec{\Omega} \times \vec{S},\ \vec{\Omega} = 2\left(\mathbf{A}\dot{\mathbf{p}}\right)\vec{u}_3, \qquad \text{(S26)}$$

where $\vec{u}_3$ is the basis vector of the S_3 axis. Thus, the Stokes vector precesses about the S_3 axis on the Poincaré sphere with the angular velocity $\Omega = 2\mathbf{A}\dot{\mathbf{p}}$, Fig. 2B. In the ray-accompanying coordinate frame attached to the Frenet trihedron, $\left(\mathbf{t}, \mathbf{v}, \mathbf{w}\right) = \left(\mathbf{t}, \mathbf{n}, \mathbf{b}\right)$, we have[S10] $\mathbf{A}\dot{\mathbf{p}} \to -T^{-1}$. Therefore, one period of the helical ray in Fig. 1A causes azymuthal rotation of the Stokes vector on the angle[33]

$$2\Phi_B = -2\int T^{-1} dl = -4\pi + 2\Theta \qquad \text{(S27)}$$

on the Poincaré sphere, Fig. 2B, where Θ is the solid angle enclosed by the trajectory of the $\mathbf{t}$ vector on the unit sphere in momentum space, Fig. 2A. The factor of 2 occurs in the evolution of the Stokes vector because a complete 2π turn of the polarization ellipse in the real space corresponds to a 4π double-turn on the Poincaré sphere.

The above two forms of the equation of the polarization evolution, Eqs. (S23) and (S26), i.e., the Jones and Stokes representations, are the optical counterparts of the Schrödinger and Heisenberg pictures of spin-1/2 evolution in quantum mechanics[19,S11,S12]. Of course, the spin of a photon is 1, but the Stokes vector is rather a *pseudo*-spin in the problem with two polarization modes.

3. Modified theory for the light propagation along a reflecting surface

Let us consider a number of total internal reflections of a light beam at a nearly grazing angle from a concave dielectric interface, Fig. S1. In the limit of zero angle $\alpha \to 0$ the distance between two successive reflections $\Delta l \to 0$, and the beam propagates along the smooth surface. Using the geometrical features of the reflection, which is symmetric with respect to the normal to the surface, $\mathbf{N}$, lying in the plane of propagation, we conclude that the normal to the sliding ray coincides with the normal to the surface: $\mathbf{n} = \mathbf{N}$. As a consequence, the radii of curvature of the ray, R, and the surface cross-section including the ray tangent $\mathbf{t}$, R_N, coincide: $R_N = R$. The Frenet-Serret formula for the evolution of the tangent $\mathbf{t}$ reads $\dot{\mathbf{t}} = \dfrac{\mathbf{n}}{R}$. Substituting here parameters of the ray with the parameters of the surface and using $\mathbf{t} = \mathbf{p}/p$ with $p = \text{const}$ in the homogeneous dielectric medium, we arrive at

$$\frac{\dot{\mathbf{p}}}{p} = \frac{\mathbf{N}}{R_N}. \tag{S28}$$

This is the first modified equation of motion Eq. (6).

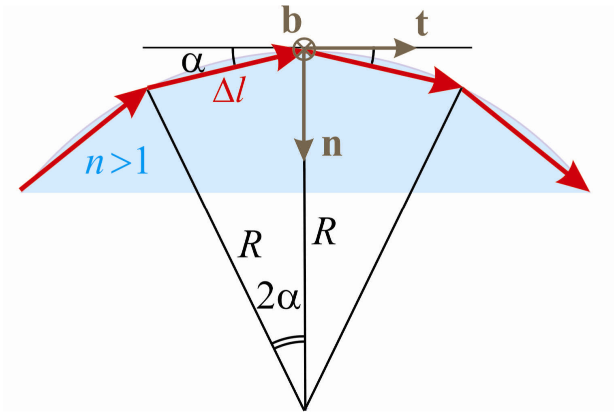

Fig. S1. Geometry of the successive total internal reflections from an element of a concave surface at a nearly grazing angle.

In order to show that the second Eq. (6) is valid, we consider the Imbert-Fedorov transverse shift $\Delta\mathbf{r}_{IF}$ at a single total internal reflection (see Refs. 14,15 and references therein), Fig. S1. The transverse shift out of the propagation plane is directed along the binormal $\mathbf{b}$ to the ray. Using the formula obtained in Ref. 15 and expressing it in terms of Stokes parameters, the transverse shift at a total internal reflection can be written as

$$\Delta\mathbf{r}_{IF} = -\lambda\!\!\!^{-} \tan\alpha \left[S_3 \left(1 + \mathrm{Re}\frac{\rho_\perp}{\rho_\parallel} \right) + S_2 \,\mathrm{Im}\frac{\rho_\perp}{\rho_\parallel} \right] \mathbf{b}. \tag{S29}$$

Here $\rho_\parallel$ and $\rho_\perp$ are the Fresnel reflection coefficients for the waves linearly polarized, respectively, along $\mathbf{n}$ and $\mathbf{b}$[36]. In the limit $\alpha \to 0$ we have

$$\rho_\perp / \rho_\parallel \approx 1 + 2i\alpha\sqrt{1 - n^{-2}} \tag{S30}$$

and

$$\Delta\mathbf{r}_{IF} \approx -2\lambda\alpha S_3 \mathbf{b}\,. \tag{S31}$$

The ray length between two successive reflections equals $\Delta l \approx 2\alpha R$. Hence, in the limit $\alpha \to 0$ the Imbert-Fedorov shift per unit ray length, $\dfrac{\Delta\mathbf{r}_{IF}}{\Delta l} \to \delta\dot{\mathbf{r}}$, leads to the differential equation

$$\delta\dot{\mathbf{r}} = -\lambda S_3 R^{-1}\mathbf{b}\,. \tag{S32}$$

Taking into account that $R^{-1}\mathbf{b} = -\mathbf{p}\times\dot{\mathbf{p}}/p^2$ and $\lambda = \lambda_0/p$, Eq. (S32) gives precisely the topological term in the second Eq. (6).

Finally, we aim to derive the modified equation for the polarization evolution along the surface. Equation (S30) implies that the p and s linearly polarized modes acquire the phase difference as the wave is reflected from the surface. This phase difference equals

$$\Delta\Phi = \Phi_{\mathbf{n}} - \Phi_{\mathbf{b}} \approx -2\alpha\sqrt{1-n^{-2}}\,, \tag{S33}$$

where $\Phi_{\mathbf{n}}$ and $\Phi_{\mathbf{b}}$ are the phase acquired by the p and s modes (which are polarized along $\mathbf{n}$ and $\mathbf{b}$. Thus, in the limit $\alpha \to 0$, the phase difference per unit ray length yields $\dfrac{\Delta\Phi}{\Delta l} \to \delta\dot{\Phi} = -R^{-1}\sqrt{1-n^{-2}}$. This phase difference is described by the following differential equation for the Jones vector in the basis of linear polarizations:

$$\frac{d}{dl}\begin{pmatrix} E_{\mathbf{n}} \\ E_{\mathbf{b}} \end{pmatrix} = -i\frac{R^{-1}\sqrt{1-n^{-2}}}{2}\hat{\sigma}_3\begin{pmatrix} E_{\mathbf{n}} \\ E_{\mathbf{b}} \end{pmatrix}, \tag{S34}$$

where $E_{\mathbf{n}}$ and $E_{\mathbf{b}}$ are the electric field projections on $\mathbf{n}$ and $\mathbf{b}$. By performing transformation to the basis of circularly polarized modes $E^{\pm} = E_{\mathbf{n}} \mp iE_{\mathbf{b}}$ (cf. Eqs. (S3), (S7), and (S20)) and adding the Berry phase term, Eq. (S23), we arrive at the following equation for the polarization evolution:

$$\left|\dot{\psi}\right\rangle = -i\left[-T^{-1}\hat{\sigma}_3 + \frac{R^{-1}\sqrt{1-n^{-2}}}{2}\hat{\sigma}_1\right]\left|\psi\right\rangle, \tag{S35}$$

where we took into account that in the Frenet coordinate frame the Berry term $-\mathbf{A}\dot{\mathbf{p}}$ takes the form of the ray torsion T^{-1}.

Similarly to Eqs. (S23), (S25), and (S26), the transition to the Stokes-vector representation leads to the equation

$$\dot{\vec{S}} = \vec{\Omega}\times\vec{S}, \quad \vec{\Omega} = \left(\sqrt{1-n^{-2}}R^{-1}, 0, -2T^{-1}\right), \tag{S36}$$

which is the equation of motion (7). Eqs. (S35) and (S36) are of the form of the polarization evolution equations in an anisotropic medium with a linear birefringence due to the curvature term and a circular birefringence due to the torsion (Berry phase) term (see, e.g., Refs. S11–S16). In the Jones representation, the polarization evolution equation for a curved rays in an anisotropic medium was obtained by Kravtsov[S17,S18], while in the Stokes vector representation it was recently derived in Refs. 19,S12,S19. It should be noted that unlike the quadratic effect of the ray curvature in an isotropic medium[30,33,S20,S21], the effective linear birefringence in Eqs. (S35) and (S36) is of the *first* order in the curvature R^{-1}.

4. Calculations for a helical ray trajectory inside a dielectric cylinder

Here we integrate the equations of motion (6) and (7) for a helical ray trajectory with a constant curvature and torsion. The solution of Eq. (7) for the output Stokes vector $\vec{S}(l)$ is expressed by the Rodrigues formula[S22] for the rotation of the initial Stokes vector $\vec{S}_{\text{in}}$ on the angle Ωl about $\vec{\omega} = \vec{\Omega}/\Omega$:

$$\vec{S}_{\text{out}} = \vec{S}_{\text{in}}\cos(\Omega l) + \left(\vec{\omega}\times\vec{S}_{\text{in}}\right)\sin(\Omega l) + \left(\vec{\omega}\vec{S}_{\text{in}}\right)\vec{\omega}\left[1-\cos(\Omega l)\right]. \tag{S37}$$

For a right- and left-hand circularly polarized incident wave, $\vec{S}_{\rm in}^{(R,L)} = (0,0,\pm 1)$, Eq. (S37) yields Eq. (8):

$$\vec{S}^{(R,L)} = \pm\Big(\omega_1\omega_3\big[1-\cos(\Omega l)\big], -\omega_1 \sin(\Omega l), \big(1-\omega_3^2\big)\cos(\Omega l)+\omega_3^2\Big). \qquad \text{(S38)}$$

The trajectory displacement $\delta\mathbf{r}$ is obtained by integration of Eq. (S32).

$$\delta\mathbf{r} = -\lambda\!\!\!^{-} R^{-1}\mathbf{b}\int_0^l S_3\, dl = -\lambda\!\!\!^{-} R^{-1} l\mathbf{b}\langle S_3\rangle . \qquad \text{(S39)}$$

Here $\langle S_3 \rangle = \frac{1}{l}\int_0^l S_3\, dl$ is the averaged value of the wave helicity on the ray trajectory. Substituting Eq. (S38), we obtain

$$\Big\langle S_3^{(R,L)} \Big\rangle = \pm\left[\omega_3^2 + \big(1-\omega_3^2\big)\frac{\sin(\Omega l)}{\Omega l}\right]. \qquad \text{(S40)}$$

Equation (S39) with (S40) gives Eq. (9) for the trajectory displacement. For a helical ray with radius R_0 and angle of propagation θ, the ray curvature and torsion are equal, respectively, to $R^{-1} = R_0^{-1}\sin^2\theta$ and $T^{-1} = R_0^{-1}\sin\theta\cos\theta$, which should be substituted in the above equations (S36)–(S40).

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S3. The Coriolis term in Eq. (S9), arising due to the rotation of the reference frame with respect to the medium, can also be treated as a rotational Doppler shift, see in L. Allen, S. M. Barnett, and M. Padgett, *Optical Angular Momentum* (IOP Publishing, Bristol, 2003). The medium rotates with respect to the reference frame with the opposite angular velocity $-\mathbf{\Lambda}$, which induces the corresponding rotational Doppler shift associated with the angular momentum carried by the light.

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S10. This is clear from comparison of the results of (*28–34*). One cannot write $\mathbf{A}(\mathbf{p})\dot{\mathbf{p}} = -T^{-1}$ because the Frenet trihedron is parametrized not by $\mathbf{p}$ only, but also by its derivative $\dot{\mathbf{p}}$ and $T = T(\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}})$. Nonetheless, all the equations remain true in the Frenet frame assuming substitution $\mathbf{A}(\mathbf{p})\dot{\mathbf{p}} \to -T^{-1}$.

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